

THE LAMB-BATEMAN INTEGRAL EQUATION AND THE FRACTIONAL DERIVATIVES

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ABSTRACT. The Lamb-Bateman integral equation was introduced to study the solitary wave diffraction and its solution was written in terms of an integral transform. We prove that it is essentially the Abel integral equation and its solution can be obtained using the formalism of fractional calculus.

This letter is motivated by an interesting article [1] describing the life and achievements of Harry Bateman. The article reports, among the other things, the following integral equation proposed by Lamb [2] in his analysis of the diffraction of a solitary wave

$$(1) \quad \int_0^\infty dy u(x - y^2) = f(x) ,$$

where $u(x)$ is a function to be determined. Either refs. [1, 2] reports the solution suggested by Bateman without any proof. Here we show that relevant solution can be obtained using an operational formalism involving the fractional derivatives.

By using the identity

$$(2) \quad e^{\lambda \partial_x} g(x) = g(x + \lambda)$$

eq. (1) can be re-written as follows

$$(3) \quad \hat{O} u(x) = f(x) ,$$

where \hat{O} is the following operator

$$(4) \quad \hat{O} = \int_0^\infty dy e^{-y^2 \partial_x} .$$

The use of the Gaussian integral

$$(5) \quad \int_0^\infty dz e^{-az^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$$

allows to write

$$(6) \quad \hat{O} = \frac{1}{2} \sqrt{\frac{\pi}{\partial_x}}$$

provided that we assume the validity of the Gaussian integral for a replaced by an operator, and, therefore, the solution of eq. (1) can be written as the derivative of order $1/2$ of the function $f(x)$, namely

$$(7) \quad u(x) = \frac{2}{\sqrt{\pi}} \partial^{1/2} f(x) .$$

The theory of fractional calculus [3] allows the evaluation of the so called *difference integral* by means of an integral transform, namely

$$(8) \quad \partial^{-\mu} g(x) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^x d\xi g(\xi) (x - \xi)^{\mu-1} ,$$

which in the case $\mu = 1/2$, yields

$$(9) \quad \partial^{-1/2} g(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^x d\xi \frac{g(\xi)}{\sqrt{x - \xi}} .$$

Accordingly, we recast eq. (7) in the form

$$(10) \quad u(x) = \frac{2}{\sqrt{\pi}} \partial^{-1/2} [\partial_x f(x)] ,$$

and, thus¹

$$(11) \quad u(x) = \frac{2}{\pi} \int_{-x}^{\infty} d\xi \frac{f'(-\xi)}{\sqrt{x + \xi}} .$$

which is the solution proposed by Bateman (see refs. [1, 2] for further comments).

Since this solution has been reported without any comment, it is not clear what procedure Bateman followed. Lamb in his paper reported a different solution in the form of a double integral, but added the remark: *Mr. H. Bateman, to whom I submitted the question, has obtained a simpler solution.* Nowadays the operational solution and the use of fractional derivatives looks natural, but it is not clear if these methods were used at the time Lamb wrote his paper.

The procedure here outlined can also be exploited for integral equations that generalizes eq. (1), as, for example

$$(12) \quad \int_0^{\infty} dy u(x - y^m) = f(x) .$$

We write, indeed

$$(13) \quad \hat{O}_m u(x) = f(x) ,$$

where

$$(14) \quad \hat{O}_m = \int_0^{\infty} dy e^{-y^m \partial_x} = \frac{1}{m} \Gamma\left(\frac{1}{m}\right) \partial_x^{-1/m} ,$$

and the solution of eq. (12) is given by

$$(15) \quad u(x) = \frac{m}{\Gamma\left(\frac{1}{m}\right)} \partial_x^{1/m} f(x) ,$$

which, finally, yields²

$$(16) \quad u(x) = \text{sinc}\left(\frac{\pi}{m}\right) \int_{-x}^{\infty} d\xi \frac{f'(-\xi)}{(x + \xi)^{1/m}} .$$

Furthermore, the method can be extended to the study of the integral equation

$$(17) \quad \int_{-\infty}^{\infty} dy u(x - ay^2 + by) = f(x) .$$

¹The change of variable is not necessary and was performed only to get the same expression given by Bateman. As for the notation, we put $f'(-\xi) = \partial_y f(y)|_{y=-\xi}$.

²We used the Euler reflection formula: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$.

In this case the operator \hat{O} is

$$(18) \quad \hat{O} = \int_0^\infty dy e^{-y^2 \hat{A} + y \hat{B}} = \sqrt{\frac{\pi}{\hat{A}}} \exp\left(\frac{\hat{B}^2}{4\hat{A}}\right)$$

where $\hat{A} = a \partial_x$ and $\hat{B} = b \partial_x$. Therefore, we obtain

$$(19) \quad u(x) = \frac{\sqrt{a}}{\pi} \int_{-\infty}^x d\xi \frac{f'(\xi - c)}{\sqrt{x - \xi}}, \quad \left(c = \frac{b^2}{4a}\right).$$

We close this letter noticing that if we set $x - y^2 = z$ in eq. (1), we obtain

$$(20) \quad \int_0^\infty dy u(x - y^2) = \frac{1}{2} \int_{-\infty}^x dz \frac{u(z)}{\sqrt{x - z}} = f(x),$$

that is the equation introduced by Abel in his study of tautochrone problem [4], and that paved the way to the theory of integral equations.

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